

Kinematics analysis of the swash plate mechanism

S Alaci¹, R D Pentiuc¹, F-C Ciornei¹, F Buium² and O T Rusu¹

¹“Stefan cel Mare” University, Suceava, Romania

²“Gheorghe Asachi” Technical University, Iasi, Romania

stelian.alaci@usm.ro

Abstract. The swash plate mechanism is one of the most exploited mechanisms from technical applications due to its constrictive simplicity. The kinematical analysis supposes establishing the motion of the plunger as function of the motion of the plate and of the constructive characteristics of the mechanism. The work presents the kinematical analysis of the mechanism performed by two methods. In the first way, the mechanism is replaced by a mechanism with lower pairs to which the Hartenberg-Denavit procedure is applied. In the second way, the definition conditions of the pair of first class, plunger-plate, are used. The results are obviously identical for the two methods but for the second approach, the labour required for the analysis is significantly reduced. The main outcome of the paper refers to the scheme of kinematical solving of a mechanism. The type of pairs should be a criterion and for the higher pairs, the use of direct condition of the structural definition is recommended.

1. Introduction

The swash plate mechanism consists in a driving element, made from a rotating shaft with an attached plane plate and a driven element, linked to the ground via a translational pair, as shown in figure 1. The motion is transmitted using a kinematical pair of class I (sphere-plane). The above description attests that the mechanism corresponds to the definition given by Duca [1] for a cam mechanism: "the mechanism that transmits the motion from a driving element to a driven element by means of a higher pair". The contact stresses developed between the plate and the spherical tip of the plunger produces wear and in order to reduce it, an intermediate part that makes spherical pairs both with the end of the plunger and with the plane plate, is interposed [2]. This operation increases the degree of mobility of the mechanism with one unit, fact that is explained by the presence of a passive motion – the rotation of the part about its axis of symmetry. The kinematical analysis of the mechanism assumes finding the dependency between the displacement of the driven element and the rotation of the driving element.

2. Kinematics analysis using the replacing mechanism

Hartenberg and Denavit [3] proposed a general method for kinematics analysis of spatial mechanisms named the method of homogenous operators, based on the transformation of the coordinates of a point when these are changed from one reference system to another one.

$$\begin{bmatrix} \mathbf{x}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{12} & \mathbf{d}_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_2 \\ 1 \end{bmatrix} \quad (1)$$

The position vectors of the point in the two coordinate systems have the column matrices $\mathbf{x}_{1,2}$, the vector describing the displacement necessary to overlap the origin of the system 1 over the origin of the system 2 is $\mathbf{d}_{12}$ and the rotation matrix that superposes the axes of the system 1 over the axes of system 2 is $\mathbf{R}_{12}$, with the columns as the projections of the versors of system 2 on the axis of system 1.



Figure 1. The swash plate mechanism

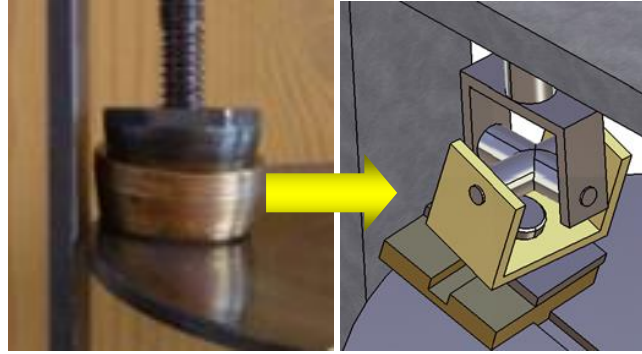


Figure 2. The sphere-plane pair replaced by the Oldham and Cardan clutches

The equation (1) can be expressed in concentrated manner:

$$\mathbf{r}_1 = \mathbf{T}_{12}\mathbf{r}_2 \quad (2)$$

Next, a closed contour made by the elements of a mechanism is considered and the convention that the " $n + 1$ "-th element coincides to the first element " 1 ". The consequence of relation (2) is:

$$\mathbf{r}_1 = \mathbf{T}_{12}\mathbf{r}_2 = \mathbf{T}_{12}\mathbf{T}_{23}\mathbf{r}_3 = \dots = \mathbf{T}_{12}\mathbf{T}_{23}\dots\mathbf{T}_{n-1,n}\mathbf{r}_n = \mathbf{T}_{12}\mathbf{T}_{23}\dots\mathbf{T}_{n-1,n}\mathbf{T}_{n,1}\mathbf{r}_1 \quad (3)$$

From equations (3) it results:

$$\mathbf{T}_{12}\mathbf{T}_{23}\dots\mathbf{T}_{n-1,n}\mathbf{T}_{n,1} = \mathbf{I}_4 \quad (4)$$

Hartenberg and Denavit demonstrated [3] that when the structure of the kinematic chain contains only cylindrical pairs (with particular cases of rotation or translation), the relative position of two systems attached respectively to two neighbouring elements necessitates only four parameters for describing it instead of six, as in the general case. To this end, the axes of cylindrical pairs play the part of " z " axes while the " x " axes will be represented by the common normals of the neighbouring " z " axes. The displacement of the current frame " k " over the next frame, " $k + 1$ " is accomplished by two rototranslations, the first along the " z " axis, characterized by the angle θ_k and the translation s_k followed by a rototranslation about the $x_{k,k+1}$ axis (which is the common normal for the axes " z_k " and " z_{k+1} ") with the angle $\alpha_{k,k+1}$ and translation $a_{k,k+1}$. One can write:

$$\mathbf{T}_{k,k+1} = \mathbf{Z}(\theta_k, s_k)\mathbf{X}(\alpha_{k,k+1}, a_{k,k+1}) \quad (5)$$

where:

$$\mathbf{Z}(\theta, s) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{X}(\alpha, a) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

Therefore, the wobble plate mechanism should be replaced by an equivalent mechanism containing in its structure only cylindrical pairs. Taking into account the fact that the higher pair of first class cancels only the translation along the common normal in the contact point, it results that the simple relative motions from the sphere-plane contact are two translations along two normal directions from the plane of the plate and three rotations performed about three axes reciprocally perpendicular. Figure 2 presents one of the possible replacements of the higher pair. It can be noticed that the substitution was made using an Oldham clutch and a Cardan clutch, linked between a rotational pair. The mechanism that substitutes the swash plate mechanism is presented in figure 3.

$$\mathbf{Z}(\theta_1, 0) \mathbf{X}(\pi/2 + \alpha,) \mathbf{Z}(\pi/2, s_2) \mathbf{X}(\pi/2, 0) \mathbf{Z}(\pi/2, s_3) \mathbf{X}(\pi/2, 0) \mathbf{Z}(\theta_4, 0) \mathbf{X}(\pi/2, 0) \cdot \mathbf{Z}(\theta_5, 0) \mathbf{X}(\pi/2,) \mathbf{Z}(\theta_6, 0) \mathbf{X}(\pi/2, 0) \mathbf{Z}(\pi/2, s_7) \mathbf{X}(0, d) = \mathbf{I}_4 \quad (7)$$

The unknowns from the equation (7) are the translations from the pairs 2, 3 and 7 and the rotations from the pairs 4, 5 and 6. McCarthy specifies [4] that the use of the closing equation under this form conducts to complex calculus and the equation must be rewritten by decomposing the left member into two factors. For the present case, the equation was rewritten as:

$$\mathbf{Z}(\theta_1, 0) \mathbf{X}(\pi/2 + \alpha,) \mathbf{Z}(\pi/2, s_2) \mathbf{X}(\pi/2, 0) \mathbf{Z}(\pi/2, s_3) \mathbf{X}(\pi/2, 0) \mathbf{Z}(\theta_4, 0) \mathbf{X}(\pi/2, 0) = [\mathbf{Z}(\theta_5, 0) \mathbf{X}(\pi/2, 0) \mathbf{Z}(\theta_6, 0) \mathbf{X}(\pi/2, 0) \mathbf{Z}(\pi/2, s_7) \mathbf{X}(0, d)]^{-1} \quad (8)$$

Considering the identity $(\mathbf{AB})^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1}$, the equation (8) can be amended as:

$$\mathbf{Z}(\theta_1, 0) \mathbf{X}(\pi/2 + \alpha,) \mathbf{Z}(\pi/2, s_2) \mathbf{X}(\pi/2, 0) \mathbf{Z}(\pi/2, s_3) \mathbf{X}(\pi/2, 0) \mathbf{Z}(\theta_4, 0) \mathbf{X}(\pi/2, 0) = \mathbf{X}(0, d)^{-1} \mathbf{Z}(\pi/2, s_7)^{-1} \mathbf{X}(\pi/2, 0)^{-1} \mathbf{Z}(\theta_6, 0)^{-1} \mathbf{X}(\pi/2, 0)^{-1} \mathbf{Z}(\theta_5, 0)^{-1} \quad (9)$$

A second remark that can reduce considerably the time for calculus comes from Uicker [5] who stipulates that the inverse of any of the operators $\mathbf{Z}(\theta, s)$ or $\mathbf{X}(\alpha, a)$ is obtained by changing the signs of the two scalar parameters from the definition relations. The final form of the relation (9) becomes:

$$\mathbf{Z}(\theta_1, 0) \mathbf{X}(\pi/2 + \alpha,) \mathbf{Z}(\pi/2, s_2) \mathbf{X}(\pi/2, 0) \mathbf{Z}(\pi/2, s_3) \mathbf{X}(\pi/2, 0) \mathbf{Z}(\theta_4, 0) \mathbf{X}(\pi/2, 0) = \mathbf{X}(0, -d) \mathbf{Z}(-\pi/2, -s_7) \mathbf{X}(-\pi/2, 0) \mathbf{Z}(-\theta_6, 0) \mathbf{X}(-\pi/2, 0) \mathbf{Z}(-\theta_5, 0) \quad (10)$$

$$\begin{bmatrix} -\sin \theta_1 \cos \alpha \cos \theta_4 - \cos \theta_1 \sin \theta_4 & -\sin \theta_1 \sin \alpha & -\sin \theta_1 \cos \alpha \sin \theta_4 + \cos \theta_1 \cos \theta_4 & s_3 \cos \theta_1 + s_2 \cos \alpha \sin \theta_1 \\ \cos \theta_1 \cos \alpha \cos \theta_4 - \sin \theta_1 \sin \theta_4 & \cos \theta_1 \sin \alpha & \cos \theta_1 \cos \alpha \sin \theta_4 + \sin \theta_1 \cos \theta_4 & s_3 \cos \theta_1 - s_2 \cos \alpha \cos \theta_1 \\ \sin \alpha \cos \alpha & -\cos \alpha & \sin \alpha \sin \theta_4 & -s_2 \sin \alpha \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sin \theta_5 & -\cos \theta_5 & 0 & -d \\ -\cos \theta_5 \cos \theta_6 & -\sin \theta_5 \cos \theta_6 & -\sin \theta_6 & 0 \\ \cos \theta_5 \sin \theta_6 & \sin \theta_5 \sin \theta_6 & -\cos \theta_6 & -s_7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (11)$$

From equation (11) it can be remarked that the information on the translations occur in the last column of the matrices. By equating the corresponding elements of the last columns, it results:

$$\begin{cases} s_3 \cos \theta_1 + s_2 \cos \alpha \sin \theta_1 = -d \\ s_3 \cos \theta_1 - s_2 \cos \alpha \cos \theta_1 \\ -s_2 \sin \alpha = -s_7 \end{cases} \quad (12)$$

The solutions of the system are immediately obtained:

$$s_2(\theta_1) = -d \sin \theta_1 / \cos \alpha ; \quad s_3(\theta_1) = -d \cos \theta_1 ; \quad s_7(\theta_1) = -d \tan \alpha \sin \theta_1 \quad (13)$$

The first two relations can be regarded as the parametric equations of the trajectory of the contact point and represent an ellipse having the semi-axes $d / \cos \alpha$ and d . An interesting aspect can be noticed from figure 4: the trajectory of the contact point is actually an ellipse but it is eccentric with respect to the circular plate. This fact is produced by the spherical surface from the end of the plunger. The plate and the plunger in the extreme positions are presented in figure 5.

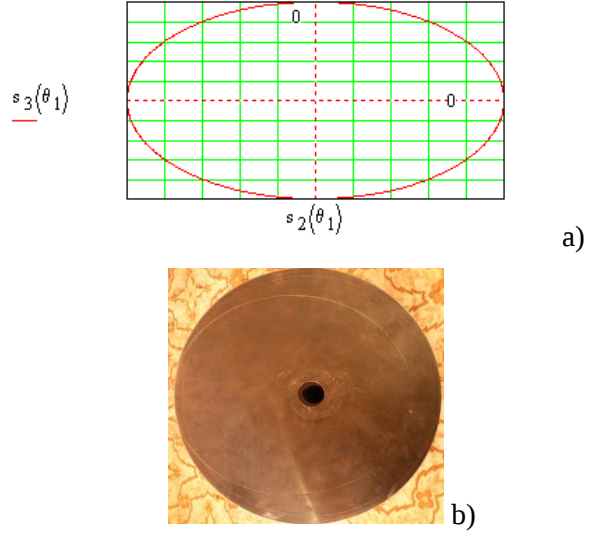
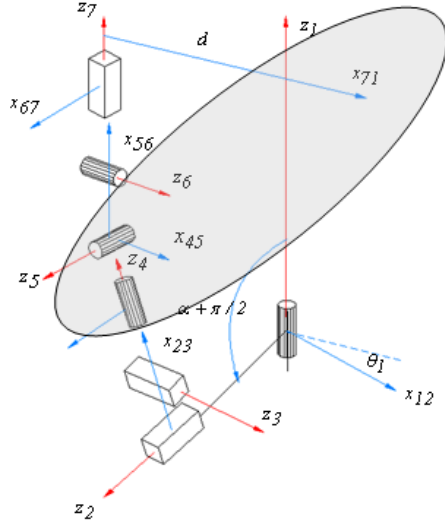


Figure 3. The mechanism that replaces the swash plate mechanism

Figure 4. The trajectory of the contact point: a) model; b) experimental trace

For this arrangement of the plate, the angle α between the normal to the plate and the rotation axis of the main shaft is seen in its actual size. The points $T_{1,2}$ correspond to the contact points between the plate and the plunger when the ball radius is zero and are placed at the distance $d / \cos \alpha$ with respect to the axis of rotation of the plate, in agreement to the remarks based on the relations (13). When a ball exists at the end of the plunger, the contact points move from $T_{1,2}$ in the positions $C_{1,2}$, and therefore the elliptical trajectory "ascends" on the plate with the distance $R \tan \alpha$.

3. The direct kinematical analysis using the constraint equation of the higher pair

The law of motion of the plunger was found after replacing the pair of first class between the plate and the plunger and using the equivalent mechanism. The drawback of the method consists in the large amount of calculus required for establishing the motion in all the pairs of the replacing chain (part of them not being necessary). When the structure of the mechanism contains pairs of first or second class, it is convenient to perform the analysis on the actual mechanism directly, because the small number of constraints imposed by those pairs require a reduced number of equations of condition. For the actual case, the equation of condition obliges that the tip of the plunger always is contained in the plane of the plate. The current position of the normal to the plane of the plate, after a plate rotation of θ angle about the vertical of versor $\mathbf{k}$, shown in figure 6, is, according to Angeles [6]:

$$\mathbf{n} = [\mathbf{k}\mathbf{k}^T + (\mathbf{I}_3 - \mathbf{k}\mathbf{k}^T) \cos \theta + \tilde{\mathbf{k}} \sin \theta] \mathbf{n}_0 \quad (14)$$

where $\mathbf{n}_0$ is the versor of the plane of the plate for the initial position $\theta = 0$.

$$\mathbf{k} = [0 \ 0 \ 1]^T; \quad \mathbf{n}_0 = [\sin \alpha \ 0 \ \cos \alpha]^T \quad (15)$$

After the calculus is made, it results:

$$\mathbf{n} = [\sin \alpha \cos \theta \ \sin \alpha \sin \theta \ \cos \alpha]^T \quad (16)$$

The equation of the plane of the plate for a current position is:

$$\mathbf{n} \cdot \mathbf{r} = 0 \Leftrightarrow x \sin \alpha \cos \theta + y \sin \alpha \sin \theta + z \cos \alpha = 0 \quad (17)$$

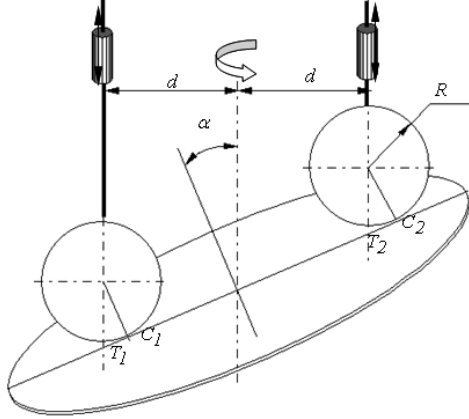


Figure 5. The plate and the plunger in the extreme positions

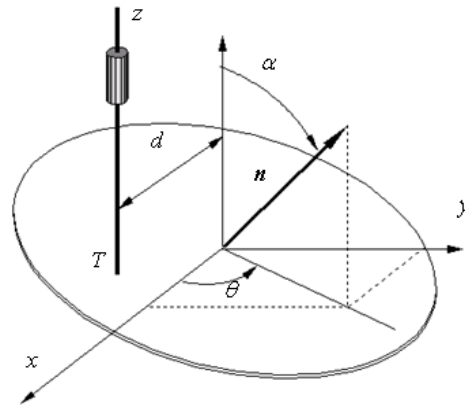


Figure 6. The normal to the plate for a current position

The equations of the direction of motion of the plunger are added:

$$x = d; \quad y = 0; \quad z = -d \tan \alpha \cos \theta \quad (18)$$

Equation (13) and equation (18), express identical harmonic motions with the amplitude $d \tan \alpha$, but with different origins for the angle of rotation of the plate.

4. Conclusions

The paper presents the kinematical analysis of the swash plate mechanism, for which two methods are considered. The first one assumes replacing the pair between the plunger and the plate with a kinematical chain, structurally equivalent, consisting only in rotational pairs and translational pairs to which the Hartenberg-Denavit method can be applied. The method has the advantage of an increased degree of algorithmization but also has the disadvantage of large calculus required for finding the motions from all the pairs of the kinematical chain of the replacing mechanism, from which some are not necessary or even don't have a physical significance. The second method consists in using directly the actual mechanism. Since the number of constraints imposed by the higher pair is reduced, only one for the present case, the equations of condition are simply to express, that is to state mathematically the fact that the tip of the plunger is always placed in the plane of the plate. The second method is more expedite than the first one but requires the identification and formulation of the accurate mathematical expressions of the condition equations for kinematical pairs.

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Acknowledgment

„This work was supported by a grant of the Romanian Ministry of Research and Innovation, CCCDI – UEFISCDI, project number PN-III-P1-1.2-PCCDI-2017-0404 / 31PCCDI/2018, within PNCDI III”.